Think-Tac-Toe: When are puzzles solvable?

We present a new pen-and-paper logic puzzle that we call Think-Tac-Toe (based on a surface similarity to Tic-Tac-Toe, although the puzzle mechanics more closely resemble the computer game Minesweeper). In Think-Tac-Toe, you are given a rectangular grid of numbers and asked to logically deduce which cells contain X's and which O's. The number in each cell represents the number of X's in that cell's neighborhood, where a cell's neighborhood is composed of the cell itself and any adjacent cells (including diagonals). Even small grids (e.g. 3x3) can provide a reasonable challenge of deductive logic, while larger grids can encode hidden pictures for the puzzler to discover. These puzzles were well received in a middle school math enrichment program that we teach. However, more interestingly, only certain grid sizes are guaranteed solvable (i.e. possess a unique solution for any configuration of X's and O's). We show that a grid size is guaranteed solvable if and only if the adjacency matrix of the graph associated with the connections among grid cells is invertible. For any given grid size, this can be found using basic linear algebra. We further prove that a grid size is guaranteed solvable if and only if both dimensions are not congruent to 2 (modulo 3). However, it is an open question to characterize when a puzzle will be guaranteed solvable when played on an arbitrary graph (rather than a rectangular grid). We will distribute example Think-Tac-Toe puzzles to the audience, and discuss the underlying mathematics.